Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 291 (2004) 341–351

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping

Yuan Gong Sun

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China

Received 21 April 2003

Submitted by J.S.W. Wong

Abstract

Some new oscillation criteria are established for the nonlinear damped differential equation $(r(t)y')' + p(t)y' + q(t)f(y) = 0$ that are different from most known ones in the sense that they are based on a class of new functions $\Phi(t, s, r)$ defined in the sequel. Our results are sharper than some previous results which can be seen by the examples at the end of this paper.

Keywords: Second order; Damping; Oscillation

1. Introduction

We consider the oscillatory behavior of solutions of the second-order nonlinear damped differential equation

$$\big(r(t)y'\big)' + p(t)y' + q(t)f(y) = 0 \tag{1.1}$$

on the half line $[t_0, \infty)$. In Eq. (1.1) we assume that $r(t) \in C([t_0, \infty), (0, \infty))$, $p(t),\ q(t) \in C([t_0, \infty), R)$, $f(u) \in C(R, R)$, $uf(u) > 0$, and $f'(u) \geqslant \mu > 0$ for $u \neq 0$, where $R = (-\infty, \infty)$.

We recall that a function $y : [t_0, t_1) \to R$, $t_1 > t_0$ is called a solution of Eq. (1.1) if $y(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In the sequel, it will be always assumed that solutions

E-mail address: yugsun@263.net.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.11.008

of Eq. (1.1) exist on some half-line $[T, \infty)$ $(T \geqslant t_0)$. A solution $y(t)$ of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for different classes of second-order differential equations [1–20]. In the absence of damping, there are a great number of papers (see, for example, [1,2,9–12,14–17] and the references therein) devoted to the particular cases of Eq. (1.1) such as

$$y'' + q(t)y = 0, \tag{1.2}$$

$$\big(r(t)y'\big)' + q(t)y = 0, \tag{1.3}$$

and

$$\big(r(t)y'\big)' + q(t)f(y) = 0. \tag{1.4}$$

An important tool in the study of oscillatory behavior of solutions for the type of Eq. (1.1) is the averaging technique. This goes back as far as the classical results of Winter [9] giving a sufficient condition for the oscillation of Eq. (1.2), namely,

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\,dv\,ds = \infty, \tag{1.5}$$

and Hartman [17] who showed that the above limit cannot be replace by the limit superior and proved the condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\,dv\,ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\,dv\,ds \leqslant \infty \tag{1.6}$$

implies that Eq. (1.2) is oscillatory.

The result of Winter was improved by Kamenev [11] who proved that the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\,ds = \infty, \quad \text{for some } n > 2, \tag{1.7}$$

is sufficient for the oscillation of Eq. (1.2).

Recently, Kong [12] proved that

Theorem A. *Equation* (1.2) *is oscillatory provided that for each $l \geqslant t_0$, there exists $\alpha > 1$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_{l}^{t} (s-l)^{\alpha} q(s)\,ds > \frac{\alpha^2}{4(\alpha-1)} \tag{1.8}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_{l}^{t} (t-s)^{\alpha} q(s)\,ds > \frac{\alpha^2}{4(\alpha-1)}. \tag{1.9}$$

Li and Agarwal [13] extend the main results of Kong [12] to Eq. (1.1), and obtained the following results:

Theorem B. *Assume that* $\lim_{t\to\infty} R(t)=\infty$, *where* $R(t)=\int_l^t ds/r(s)$ *for* $t\geqslant l\geqslant t_0$. *Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l\geqslant t_0$ *and for some* $\lambda>1$, *the following two inequalities hold*:

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t\left\{\left(q(s)-\frac{p^2(s)}{4\mu r(s)}\right)\bigl[R(s)-R(l)\bigr]^{\lambda} + \frac{p(s)}{2\mu r(s)}\bigl[R(s)-R(l)\bigr]^{\lambda-1}\right\}ds>\frac{\lambda^2}{4\mu(\lambda-1)} \tag{1.10}$$

and

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t\left\{\left(q(s)-\frac{p^2(s)}{4\mu r(s)}\right)\bigl[R(t)-R(s)\bigr]^{\lambda} + \frac{p(s)}{2\mu r(s)}\bigl[R(t)-R(s)\bigr]^{\lambda-1}\right\}ds>\frac{\lambda^2}{4\mu(\lambda-1)}. \tag{1.11}$$

Just as we can see, most oscillation results involve the function class X. We say that a function $H=H(t,s)$ belongs to the function class X, if $H\in C(D,R_+)$, where $D=\{(t,s)\colon t_0\leqslant s\leqslant t<\infty\}$, which satisfies $H(t,t)=0$, $H(t,s)>0$ for $t>s$, and has partial derivative $\partial H/\partial s$ and $\partial H/\partial t$ on D such that

$$\frac{\partial H}{\partial t}=h_1(t,s)\sqrt{H(t,s)}\quad\text{and}\quad\frac{\partial H}{\partial s}=-h_2(t,s)\sqrt{H(t,s)}, \tag{1.12}$$

where h_1, h_2 are locally integrable with respect to t and s, respectively, in D.

In this paper, we define another function class Y. We say that a function $\Phi=\Phi(t,s,l)$ belongs to the function class Y, denoted by $\Phi\in Y$, if $\Phi\in C(E,R)$, where $E=\{(t,s,l)\colon t_0\leqslant l\leqslant s\leqslant t<\infty\}$, which satisfies $\Phi(t,t,l)=0$, $\Phi(t,l,l)=0$, $\Phi(t,s,l)\neq 0$ for $l<s<t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E.

In Sections 2 and 3, we establish some new oscillation results for Eq. (1.1) in terms of the above definition. Our results are not contained in those of Li and Agarwal [13], Zheng [18], and Wong [19]. In fact, our results are simpler than Theorems A and B in the sense that only one condition

$$\limsup_{t\to\infty} T\left[q-\frac{r}{\mu}\left(\frac{p}{2r}-\phi\right)^2; l,t\right]>0$$

is sufficient for the oscillation of all solutions of (1.1), where the operator $T[\cdot;l,t]$ is defined by

$$T[g;l,t]=\int_l^t\Phi^2(t,s,l)g(s)\,ds\quad\text{for } t\geqslant s\geqslant l\geqslant t_0 \text{ and } g(s)\in C[t_0,\infty), \tag{1.13}$$

and the function $\phi = \phi(t, s, l)$ is defined by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \phi(t,s,l)\Phi(t,s,l). \tag{1.14}$$

It is easy to verify that $T[\cdot; l, t]$ is a linear operator and satisfies

$$T[g'; l, t] = -2T[g\phi; l, t], \quad \text{for } g \in C^1[t_0, \infty). \tag{1.15}$$

2. Kamenev-type oscillation criteria

Theorem 2.1. *Equation* (1.1) *is oscillatory provided that for each $l \geqslant t_0$, there exists a function $\Phi \in Y$ such that*

$$\lim_{t\to\infty} \sup T\left[q - \frac{r}{\mu}\left(\frac{p}{2r} - \phi\right)^2; l, t\right] > 0, \tag{2.1}$$

where the operator T is defined by (1.13) *and the function $\phi = \phi(t, s, l)$ is defined by* (1.14).

Proof. Suppose to the contrary that there exists a solution $y(t)$ of (1.1) such that $y(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Define

$$w(s) = \frac{r(s)y'(s)}{f(y(s))}, \quad s \geqslant t_1. \tag{2.2}$$

From (1.1) and (2.2) we have that for $s \geqslant t_1$

$$\begin{aligned} w'(s) &= -\frac{p(s)}{r(s)}w(s) - q(s) - r(s)\frac{[y'(s)]^2 f'(y(s))}{f^2(y(s))} \\ &\leqslant -\frac{p(s)}{r(s)}w(s) - q(s) - \mu\frac{w^2(s)}{r(s)}. \end{aligned} \tag{2.3}$$

Applying $T[\cdot; t_1, t]$ $(t > t_1)$ to (2.3), we have

$$T[w'; t_1, t] \leqslant -T\left[\frac{pw}{r}; t_1, t\right] - T[q; t_1, t] - T\left[\mu\frac{w^2}{r}; t_1, t\right].$$

By (1.15) and the above inequality, we have for $t > t_1$

$$T[q; t_1, t] \leqslant 2T[w\phi; t_1, t] - T\left[\frac{pw}{r}; t_1, t\right] - T\left[\mu\frac{w^2}{r}; t_1, t\right]. \tag{2.4}$$

Completing square of w in (2.4) and noting that (1.14), we obtain

$$T\left[q - \frac{r}{\mu}\left(\frac{p}{2r} - \phi\right)^2; t_1, t\right] \leqslant 0, \quad t > t_1.$$

Taking the super limit in the above inequality, we have

$$\lim_{t\to\infty} \sup T\left[q - \frac{r}{\mu}\left(\frac{p}{2r} - \phi\right)^2; t_1, t\right] \leqslant 0,$$

which contradicts the assumption (2.1). This completes the proof of Theorem 2.1. □

If we assume that $p(t) \in C^1([t_0, \infty), R)$, note that

$$T[p\phi; l, t] = -\frac{1}{2}T[p'; l, t],$$

then we have the following corollary by Theorem 2.1.

Corollary 2.1. *Suppose that $p(t) \in C^1([t_0, \infty), R)$, then Equation (1.1) is oscillatory provided that for each $l \geqslant t_0$, there exists a function $\Phi \in Y$ such that*

$$\lim_{t\to\infty} \sup T\left[q - \frac{p^2}{4\mu r} - \frac{p'}{2\mu} - \frac{r\phi^2}{\mu}; l, t\right] > 0, \tag{2.5}$$

where the operator T is defined by (1.13) and the function $\phi = \phi(t, s, l)$ is defined by (1.14).

If we choose $\Phi(t, s, l) = \rho(s)(t-s)^{\alpha}(s-l)^{\beta}$ for $\alpha, \beta > 1/2$ and $\rho(t) \in C^1([t_0, \infty), (0, \infty))$, then we have

$$\phi(t, s, l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha + \beta)s + \alpha l}{(t-s)(s-l)}.$$

Thus by Theorem 2.1, we have the following oscillation result:

Theorem 2.2. *Equation (1.1) is oscillatory provided that for each $l \geqslant t_0$, there exist a function $\rho(t) \in C^1([t_0, \infty), R)$ and two constants $\alpha, \beta > 1/2$, such that*

$$\lim_{t\to\infty} \sup \int_l^t \rho^2(s)(t-s)^{2\alpha}(s-l)^{2\beta}\left[q(s) - \frac{r(s)}{\mu}\left(\frac{p(s)}{2r(s)} - \frac{\rho'(s)}{\rho(s)} - \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}\right)^2\right] ds > 0. \tag{2.6}$$

Let $r(t) \equiv 1$, choose $\beta = 1$ and $\rho(t) \equiv 1$ in Theorem 2.2, then we have the following interesting theorem by Theorem 2.2.

Theorem 2.3. *Equation (1.1) with $r(t) \equiv 1$ is oscillatory provided that for each $l \geqslant t_0$, there exists a constant $\alpha > 1/2$, such that*

$$\lim_{t\to\infty} \sup \frac{1}{t^{2\alpha+1}} \int_l^t (t-s)^{2\alpha}(s-l)^2\left[4\mu q(s) - p^2(s) + 4\frac{t - (1+\alpha)s + \alpha l}{(t-s)(s-l)}p(s)\right] ds > \frac{4\alpha}{(2\alpha-1)(2\alpha+1)}. \tag{2.7}$$

Proof. Noting that

$$\int_l^t (t-s)^{2\alpha-2}\big[t-(\alpha+1)s+\alpha l\big]^2\,ds = \int_l^t (t-s)^{2\alpha-2}\big[(t-s)-\alpha(s-l)\big]^2\,ds$$

$$= \int_l^t (t-s)^{2\alpha}\,ds - 2\alpha\int_l^t (t-s)^{2\alpha-1}(s-l)\,ds + \alpha^2\int_l^t (t-s)^{2\alpha-2}(s-l)^2\,ds$$

$$= \int_l^t (t-s)^{2\alpha}\,ds - \int_l^t (t-s)^{2\alpha}\,ds + \frac{2\alpha^2}{2\alpha-1}\int_l^t (t-s)^{2\alpha-1}(s-l)\,ds$$

$$= \frac{\alpha}{2\alpha-1}\int_l^t (t-s)^{2\alpha}\,ds = \frac{\alpha}{(2\alpha-1)(2\alpha+1)}(t-l)^{2\alpha+1}. \tag{2.8}$$

Thus, from (2.6) and (2.8), we have

$$4\mu\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_l^t (t-s)^{2\alpha}(s-l)^2\left[q(s)-\frac{1}{\mu}\left(\frac{p(s)}{2}-\frac{t-(\alpha+1)s+\alpha l}{(t-s)(s-l)}\right)^2\right]ds$$

$$= \limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_l^t (t-s)^{2\alpha}(s-l)^2\left[4\mu q(s)-p^2(s) + 4\frac{t-(\alpha+1)s+\alpha l}{(t-s)(s-l)}p(s)\right]ds$$

$$- 4\lim_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_l^t (t-s)^{2\alpha-2}\big[t-(\alpha+1)s+\alpha l\big]^2\,ds$$

$$= \limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_l^t (t-s)^{2\alpha}(s-l)^2\left[4\mu q(s)-p^2(s) + 4\frac{t-(\alpha+1)s+\alpha l}{(t-s)(s-l)}p(s)\right]ds$$

$$- \frac{4\alpha}{(2\alpha-1)(2\alpha+1)}. \tag{2.9}$$

From (2.7) and (2.9), we can easily obtain

$$\limsup_{t\to\infty}\int_l^t (t-s)^{2\alpha}(s-l)^2\left[q(s)-\frac{1}{\mu}\left(\frac{p(s)}{2}-\frac{t-(\alpha+1)s+\alpha l}{(t-s)(s-l)}\right)^2\right]ds > 0,$$

and hence, Eq. (1.1) with $r(t)\equiv 1$ is oscillatory by Theorem 2.2. This completes the proof of Theorem 2.3. □

By Theorem 2.3, we have the following corollary:

Corollary 2.2. *Assume that $p(t) \in C^1([t_0, \infty), R)$, then Eq.* (1.1) *with $r(t) \equiv 1$ is oscillatory provided that for each $l \geqslant t_0$, there exists a constant $\alpha > 1/2$, such that*

$$\lim_{t\to\infty} \sup \frac{1}{t^{2\alpha+1}} \int_l^t (t-s)^{2\alpha}(s-l)^2\left[4\mu q(s) - p^2(s) - 2p'(s)\right] ds > \frac{4\alpha}{(2\alpha-1)(2\alpha+1)}. \tag{2.10}$$

If we let $r(t) \equiv 1$ and $\alpha = 1$ in Theorem 2.2, similar to the proof of Theorem 2.3, we have the following theorem:

Theorem 2.4. *Equation* (1.1) *with $r(t) \equiv 1$ is oscillatory provided that for each $l \geqslant t_0$, there exists a constant $\beta > 1/2$, such that*

$$\lim_{t\to\infty} \sup \frac{1}{t^{2\beta+1}} \int_l^t (t-s)^2(s-l)^{2\beta}\left[4\mu q(s) - p^2(s) + 4\frac{\beta t - (1+\beta)s + l}{(t-s)(s-l)} p(s)\right] ds > \frac{4\beta}{(2\beta-1)(2\beta+1)}. \tag{2.11}$$

By Theorem 2.4, the following corollary is immediate:

Corollary 2.3. *Assume that $p(t) \in C^1([t_0, \infty), R)$, then Eq.* (1.1) *with $r(t) \equiv 1$ is oscillatory provided that for each $l \geqslant t_0$, there exists a constant $\beta > 1/2$, such that*

$$\lim_{t\to\infty} \sup \frac{1}{t^{2\beta+1}} \int_l^t (t-s)^2(s-l)^{2\beta}\left[4\mu q(s) - p^2(s) - 2p'(s)\right] ds > \frac{4\beta}{(2\beta-1)(2\beta+1)}. \tag{2.12}$$

Choose $\Phi(t,s,l) = \sqrt{H_1(s,l)H_2(t,s)}$, where $H_1, H_2 \in X$. According to the simple computation, we have the following theorem by Theorem 2.1.

Theorem 2.5. *Equation* (1.1) *is oscillatory provided that for each $l \geqslant t_0$, there exist two functions $H_1 \in X$ and $H_2 \in X$ such that*

$$\lim_{t\to\infty} \sup \int_l^t H_1(s,l)H_2(t,s)\left[q(s) - \frac{r(s)}{4\mu}\left(\frac{p(s)}{r(s)} - \frac{h_1(s,l)}{\sqrt{H_1(s,l)}} + \frac{h_2(t,s)}{\sqrt{H_2(t,s)}}\right)^2\right] ds > 0, \tag{2.13}$$

where $h_1,\ h_2$ are defined as the following

$$\frac{\partial H_1(s,l)}{\partial s} = h_1(s,l)\sqrt{H_1(s,l)}, \qquad \frac{\partial H_2(t,s)}{\partial s} = -h_2(t,s)\sqrt{H_2(t,s)}.$$

Remark. Theorems 2.1–2.5 are new because we introduce a new class of kernel functions $\Phi(t,s,l)$ which is basically a product $H(t,s)H(s,l)$ for a kernel $H(t,s)$ of Philos' type. On the other hand, when Eq. (1.1) reduces to Eq. (1.2) or Eq. (1.3), conditions (2.1), (2.5)–(2.7), (2.10)–(2.13) become simpler, and they are better (in many cases) than many existing oscillation conditions. For example, if we assume that all limits in Theorem A and Corollaries 2.2, 2.3 exist, then Corollaries 2.2, 2.3 imply that Eq. (1.2) is oscillatory if either (1.8) or (1.9) holds (not necessarily both hold). This is a significant improvement of Kong's results.

Example 1. In order to show the sharpness of our theorems, let us consider the following Euler differential equation

$$y'' + \frac{m}{t}y' + \frac{n}{t^2}y = 0, \quad t \geqslant t_0 > 0, \tag{2.14}$$

where m, n are constants. Using the main results of this paper, we will prove that Eq. (2.14) is oscillatory when $(m-1)^2 < 4n$. In fact, for any constant $\beta > 1/2$ and for each $l \geqslant t_0$, we have

$$\begin{aligned} &\int_l^t (t-s)^2(s-l)^{2\beta}\big[4\mu q(s) - p^2(s) - 2p'(s)\big]\,ds \\ &\quad = \int_l^t (t-s)^2(s-l)^{2\beta}\frac{4n-m^2+2m}{s^2}\,ds. \end{aligned} \tag{2.15}$$

Noting that

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t (t-s)^2(s-l)^{2\beta}\frac{1}{s^2}\,ds &= \frac{1}{\beta(2\beta-1)(2\beta+1)}\lim_{t\to\infty}\frac{(t-l)^{2\beta}}{t^{2\beta}} \\ &= \frac{1}{\beta(2\beta-1)(2\beta+1)}. \end{aligned} \tag{2.16}$$

Since $(m-1)^2 < 4n$, i.e., $4n - m^2 + 2m > 1$, we can choose an appropriate constant $\beta > 1/2$ such that $4n - m^2 + 2m > 4\beta^2$, and hence

$$\frac{4n-m^2+2m}{\beta(2\beta-1)(2\beta+1)} > \frac{4\beta}{(2\beta-1)(2\beta+1)}. \tag{2.17}$$

By conditions (2.15)–(2.17), we can easily see that condition (2.13) holds. From Corollary 2.3, we have that Eq. (2.14) is oscillatory when $(m-1)^2 < 4n$. On the other hand, if $(m-1)^2 \geqslant 4n$, evidently, Eq. (2.14) has an nonoscillatory solution

$$y(t) = t^{\frac{1-m+\sqrt{(m-1)^2-4n}}{2}}.$$

In that sense, we know that our results are sharper.

3. Interval oscillation criteria

We can easily see that Theorems 2.1–2.5 and other results in [1–5,10–20] involve the integral of the coefficients p, q and r, and hence, requires the information of the coefficients on the entire half-line $[t_0,\infty)$. It is difficult to apply them to the cases when Eq. (1.1) is "bad" on a big part of $[t_0,\infty)$, e.g., when $\int_{t_0}^{\infty} p(t)\,dt=-\infty$ or $\int_{t_0}^{\infty} q(t)\,dt=-\infty$. In this section, we will establish several interval oscillation criteria for Eq. (1.1). The results may be applied to the extreme cases such as $\int_{t_0}^{\infty} p(t)\,dt=-\infty$ or $\int_{t_0}^{\infty} q(t)\,dt=-\infty$.

Theorem 3.1. *Equation* (1.1) *is oscillatory provided that for each $A\geqslant t_0$, there exist a function $\Phi\in Y$ and two constants $b>a\geqslant A$ such that*

$$T\left[q-\frac{r}{\mu}\left(\frac{p}{2r}-\phi\right)^2;a,b\right]>0, \tag{3.1}$$

where the operator T is defined by (1.13) *and the function $\phi=\phi(b,s,a)$ is defined by* (1.14).

Proof. With the proof of Theorem 2.1, where t and l are replaced by b and a, respectively, we can easily see that every solution of Eq. (1.1) has at least one zero in (a,b), i.e., every solution of Eq. (1.1) has arbitrarily large zero on $[t_0,\infty)$. This completes the proof of Theorem 3.1. □

Similar to the discussion in Section 2, we have the following corollaries:

Corollary 3.2. *Assume that $p(t)\in C^1([t_0,\infty),R)$, then Eq.* (1.1) *is oscillatory provided that for each $A\geqslant t_0$, there exist a function $\Phi\in Y$ and two constants $b>a\geqslant A$ such that*

$$T\left[q-\frac{p^2}{4\mu r}-\frac{p'}{2\mu}-\frac{r\phi^2}{\mu};a,b\right]>0, \tag{3.2}$$

where the operator T is defined by (1.13) *and the function $\phi=\phi(b,s,a)$ is defined by* (1.14).

Corollary 3.3. *Equation* (1.1) *is oscillatory provided that for each $A\geqslant t_0$, there exist a function $\rho(t)\in C^1[a,b]$, two constants $\alpha,\ \beta>1/2$, and two constants $b>a\geqslant A$ such that*

$$\int_a^b \rho^2(s)(b-s)^{2\alpha}(s-a)^{2\beta}\left[q(s)-\frac{r(s)}{\mu}\left(\frac{p(s)}{2r(s)}-\frac{\rho'(s)}{\rho(s)}-\frac{\beta b-(\alpha+\beta)s+\alpha a}{(b-s)(s-a)}\right)^2\right]ds>0. \tag{3.3}$$

Corollary 3.4. *Equation* (1.1) *is oscillatory provided that for each $A\geqslant t_0$, there exist two functions $H_1,H_2\in X$ and two constants $b>a\geqslant A$ such that*

$$\int_a^b H_1(s,a)H_2(b,s)\left[q(s)-\frac{r(s)}{4\mu}\left(\frac{p(s)}{r(s)}-\frac{h_1(s,a)}{\sqrt{H_1(s,a)}}+\frac{h_2(b,s)}{\sqrt{H_2(b,s)}}\right)^2\right]ds$$
$$>0. \tag{3.4}$$

Example 2. Now let us consider the following differential equation

$$y''+p(t)y'+q(t)\big(y+y^3\big)=0, \quad t\geqslant t_0, \tag{3.5}$$

where

$$\sqrt{\gamma}\,p(t)=q(t)=\begin{cases}\gamma(t-3n), & 3n\leqslant t\leqslant 3n+1,\\ \gamma(-t+3n+2), & 3n+1<t\leqslant 3n+2,\\ -n(3n+3-t)(t-3n-2), & 3n+2<t\leqslant 3n+3,\end{cases}$$

where γ is a positive constant and $n\in N_0=\{0,1,2,\ldots\}$. For any constant $T\geqslant t_0$, there exists $n\in N_0$ such that $3n\geqslant T$. Let $a=3n$, $b=3n+1$, $\alpha=\beta=1$ and $\rho(t)\equiv 1$. Since $f'(y)=1+3y^2\geqslant 1=\mu$ and $r(t)\equiv 1$, then the left hand of (3.3) becomes

$$\int_0^1 (1-s)^2s^2\left[\gamma s-\left(\frac{\sqrt{\gamma}}{2}s-\frac{1-2s}{(1-s)s}\right)^2\right]ds$$
$$=\gamma\int_0^1(1-s)^2s^3\,ds-\frac{\gamma}{4}\int_0^1(1-s)^2s^4\,ds+\sqrt{\gamma}\int_0^1(1-s)s^2(1-2s)\,ds$$
$$-\int_0^1(1-2s)^2\,ds$$
$$=\frac{\gamma}{60}-\frac{\gamma}{420}-\frac{\sqrt{\gamma}}{60}-\frac{1}{3}.$$

It is easy to see that (3.3) holds for sufficiently large γ, and hence, Eq. (3.5) is oscillatory for sufficiently large γ by Corollary 3.3. However, we have $\int_{t_0}^{\infty}p(t)\,dt=\int_{t_0}^{\infty}q(t)\,dt=-\infty$.

Example 3. Consider the following differential equation

$$y''+\cos t\,y'+c\sin t\big(y+y^5\big)=0, \quad t\geqslant 0, \tag{3.6}$$

where c is a constant. For any constant $T\geqslant t_0$, there exists $n\in N_0$ such that $2n\pi\geqslant T$. Let $a=2n\pi$, $b=(2n+1)\pi$, and $\rho(t)\equiv 1$. Choose $\Phi(t,s,r)=\sqrt{|\sin(t-s)\sin(s-r)|}$, since $f'(y)=1+5y^4\geqslant 1=\mu$, then we have $\Phi(b,s,a)=\sin s$, $\Phi^2(b,s,a)\phi^2(b,s,a)=\cos^2 s$, and

$$4\mu T\left[q-\frac{p^2}{4\mu r}-\frac{p'}{2\mu}-\frac{r\phi^2}{\mu};a,b\right]$$
$$=\int_a^b \Phi^2(b,s,a)\big\{\big[4\mu q(s)-p^2(s)-2p'(s)\big]-4\phi^2(b,s,a)\big\}\,ds$$

$$= \int_0^{\pi} \left\{\sin^2 s\left[(4c+2)\sin s - \cos^2 s\right] - 4\cos^2 s\right\} ds = 4(4c+2)/3 - \pi/8 - 2\pi.$$

From Corollary 3.2, we see that Eq. (3.6) is oscillatory for $c > 51\pi/128 - 1/2$.

Acknowledgments

The author thanks the referee for his valuable suggestions on this paper.

References

[1] G.J. Butler, The oscillatory behavior of a second order nonlinear differential equations, J. Math. Anal. Appl. 57 (1977) 273–289.

[2] G.J. Butler, L.H. Erbe, A.B. Mingarelli, Riccati techniques and variational principles in oscillation theory for linear systems, Trans. Amer. Math. Soc. 303 (1987) 263–282.

[3] S.R. Grace, Oscillation criteria for second order nonlinear differential equations with damping, J. Austral. Math. Soc. Ser. A 49 (1990) 43–54.

[4] S.R. Grace, Oscillation theorems for second order nonlinear differential equations with damping, Math. Nachr. 141 (1989) 117–127.

[5] S.R. Grace, Oscillation theorems for nonlinear differential equations of second order, J. Math. Anal. Appl. 171 (1992) 220–241.

[6] S.R. Grace, B.S. Lalli, Integral averaging technique for the oscillation of second order nonlinear differential equations, J. Math. Anal. Appl. 149 (1990) 231–277.

[7] S.R. Grace, B.S. Lalli, Oscillation theorems for second order superlinear differential equations with damping, J. Austral. Math. Soc. Ser. A 53 (1992) 156–175.

[8] S.R. Grace, B.S. Lalli, C.C. Yeh, Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term, SIAM J. Math. Anal. 15 (1984) 1082–1093.

[9] P. Hartman, On nonoscillatory linear differential equations of second order, Amer. J. Math. 74 (1952) 389–400.

[10] Ch. Huang, Oscillation and nonoscillation for second order linear differential equations, J. Math. Anal. Appl. 210 (1997) 712–723.

[11] I.V. Kamenev, Integral criterion of linear differential equations of second order, Mat. Zametki 23 (1978) 249–251.

[12] Q. Kong, Interval criteria for oscillation of second order linear ordinary differential equations, J. Math. Anal. Appl. 229 (1999) 258–270.

[13] W.T. Li, R.P. Agarwal, Interval oscillation criteria for second order nonlinear differential equations with damping, Comput. Math. Appl. 40 (2000) 217–230.

[14] H.J. Li, Oscillation criteria for second order linear differential equations, J. Math. Anal. Appl. 194 (1995) 217–234.

[15] W.T. Li, J.R. Yan, An oscillation criterion for second order superlinear differential equations, Indian J. Pure Appl. Math. 28 (1997) 735–740.

[16] Ch.G. Philos, Oscillation theorems for linear differential equation of second order, Arch. Math. (Basel) 53 (1989) 483–492.

[17] A. Winter, A criterion of oscillatory stability, Quart. Appl. Math. 7 (1949) 115–117.

[18] J.S.W. Wong, On Kamenev-type oscillation theorems for second order differential equations with damping, J. Math. Anal. Appl. 258 (2001) 244–257.

[19] Zh. Zheng, Note on Wong's paper, J. Math. Anal. Appl. 274 (2002) 466–473.

[20] J.R. Yan, Oscillation theorems for second order linear differential equations with damping, Proc. Amer. Math. Soc. 98 (1986) 276–282.